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THE SOLUTION OF THE INFINITE HORIZON TRACKING PROBLEM FOR DISCRETE TIME SYSTEMS POSSESSING AN EXOGENOUS COMPONENT

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In this paper we derive an algorithm that yields, for a discrete-time system, a control minimizing a quadratic cost functional. The system considered is linear and possesses an exogenous component. The cost functional is a quadratic tracking equation over an infinite time horizon with positive semi-definite weighting matrices such that a weighted sum of these matrices is positive definite. The infinite planning horizon Minimum Variance cost criterion and the Linear Quadratic regulator are special cases. For stabilizable systems we give a characterization of the asymptotically admissible reference trajectories.

1. Introduction

In this paper we study a problem that originates from the theory of optimal economic stabilization. In this theory the central issue is to design a policy yielding some prescribed behaviour for a dynamic economic model, devoid of specific economic content. This policy should, moreover, satisfy the requirement that it is optimal in the sense that some cost functional is minimized. This kind of problem has received much attention in the past from both control engineers and economists. In economics, e.g., people like Holbrook (1972), Aoki (1973), Pindyck (1973), Chow (1975), Preston (1977), Turnovsky (1977), Maybeck (1982), Preston et al. (1982), and de Zeeuw (1984) worked on this subject, inspired by the older generation personified by, e.g., Tinbergen (1952) and Philips (1954, 1957).

Research that has been done on this subject can be roughly split into two parts. On the one hand, much research has been done on the finite planning horizon problem. This problem concerns the setting of instrument variables so that the system follows a chosen trajectory as closely as possible during a finite planning time interval. The considered system is usually described by a linear, finite-dimensional, time-invariant difference (or differential) equation possessing a deterministic exogenous component. The deterministic component is

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included since the economy of almost every country is subjected to uncontrollable foreign influences.

On the other hand, much attention has been given to the infinite planning horizon problem. As the name already suggests, this problem concerns the same design problem but now for an infinite planning time interval. In general the extension of the planning horizon to infinity in optimization problems is nontrivial. There are a number of reasons to consider this problem. We mention two of them. In the first place the infinite planning horizon problem gives rise to controllers which stabilize the closed-loop system. Therefore, the resulting behaviour of the model is less sensitive to model disturbances and modelling inaccuracies. Secondly, the infinite horizon problem can be viewed as an approximation of the finite horizon problem with a large planning horizon. Since calculation of the optimal finite horizon planning policy gets increasingly cumbersome on an expanding horizon and the infinite horizon policy just requires once the solution of one quadratic equation, the infinite planning horizon policy is easier to implement.

Almost all research on the infinite horizon problem is restricted to systems without a deterministic exogenous component. Attempts to extend this theory for models including this component were made by Chow (1975) and Maybeck (1982). They provided a solution under very restrictive conditions on the deterministic component.

In this paper we show that, under a very mild growth assumption on the deterministic variables, results obtained by Pindyck and Chow concerning the finite planning horizon Linear Quadratic tracking problem can be extended to an infinite planning horizon. We take as a starting point a slightly generalized version of the finite planning horizon problem formulated by Pindyck (1973, pp. 27–35). In that problem we extend the planning horizon to infinity and derive under some smoothness conditions the optimal control algorithm. Though our analysis probably can be extended to more general models [like Preston considers, e.g., in (1977)], our generalized version captures already both the cases treated by Chow and Maybeck. By considering a special case of our problem, i.e., the optimal controller minimizing the infinite planning Minimum Variance criterion, we directly obtain conditions under which Chow's algorithm (1975, pp. 157–160) converges.

The paper is organized as follows. In section 2 we discuss the finite planning horizon problem and its solution. Using this result we solve in section 3 the infinite planning horizon problem. Since the calculations of the optimal controller are in general rather cumbersome, we formulate in section 4 additional conditions simplifying the implementation of the algorithm. Section 5 treats the special case of the infinite planning Minimum Variance controller. Moreover, the question is raised which reference trajectories can be ultimately tracked in a stabilizable system. We give a characterization of these reference trajectories and provide a robust controller tracking any such trajectory.

2. The finite planning horizon problem statement and its solution

Consider the system described by the following linear, finite-dimensional, time-invariant difference equation:

$$y_{k+1} = Ay_k + Bu_k + Cx_k, \qquad k = 1, 2, \ldots, \tag{1}$$

where y_k is an n-dimensional output vector, u_k is an m-dimensional control vector, and x_k is a p-dimensional uncontrollable deterministic exogenous vector. The objective is to let $\{y_k\}$ track a certain *a priori* determined sequence $\{y_k^*\}$ by choosing the control variable u_k in a suitable way. We assume that $m \leq n$, that the initial values of the system are $y_0 = \bar{y}_0$, and that the value of the sequence $\{x_i, i \geq k\}$ is known before u_k has to be chosen. Furthermore, it is assumed that matrix B has full rank.

This section contains the formulation of the cost functional in which the (often conflicting) aims are expressed and the derivation of the optimal controller minimizing this cost criterion. The considered cost criterion is motivated as follows. It is assumed that the government has a reference trajectory in mind for the targets y_k as well as for the instruments u_k, denoted by y_k^* and u_k^* respectively, which it wants to track accurately. Furthermore it is assumed that the government is only interested in the deviation from a variable from its reference value. In other words, it makes no difference to the government whether the variable remains under or above its reference value. These assumptions, together with the consideration that the deviation of all variables may be not of the same importance, make the following quadratic cost functional plausible:

$$J_N = \sum_{k=0}^{N-1} \left\{ (y_k - y_k^*)^T Q (y_k - y_k^*) + (u_k - u_k^*)^T R (u_k - u_k^*) \right\}$$

$$+ (y_N - y_N^*)^T Q (y_N - y_N^*). \tag{2}$$

Here it is assumed that Q and R are symmetric semi-positive definite matrices and that $B^T Q B + R$ is positive definite. This last technical assumption is made to simplify calculation. The assumption can be relaxed but, since the consequence of such a relaxation might be that the reader loses sight of the main lines of this paper, we omit it.

The problem is to find a control sequence $u_0, \ldots, u_{N-1}$ which minimizes this cost functional J_N. Under more severe restrictions on the weighting matrices this problem was first solved by Pindyck (1973, pp. 27–35). By a reformulation of some variables and equations it is easily seen that the algorithm given by Pindyck and the algorithm in the theorem below coincide. A proof of this theorem can be found, e.g., in Preston (1977).

Theorem 1. The optimal control sequence for system (1) minimizing J_N is given by

$$u_{k,N} = -G_{k,N}y_k - g_{k,N}, \tag{3a}$$

where

$$G_{k,N} = \left(R + B^T K_{k+1,N} B\right)^{-1} B^T K_{k+1,N} A, \tag{3b}$$

$$g_{k,N} = \left(R + B^T K_{k+1,N} B\right)^{-1}\left\{-B^T h_{k+1,N} + B^T K_{k+1,N} C x_k - R u_k^*\right\}, \tag{3c}$$

and $K_{k,N}$, respectively $h_{k,N}$, are given by the following recursive equations:

$$K_{k-1,N} = Q + A^T\left\{K_{k,N} B\left(R + B^T K_{k,N} B\right)^{-1} B^T K_{k,N}\right\}A, \tag{3d}$$

$$K_{N,N} = Q,$$

$$h_{k,N} = \left(A - BG_{k,N}\right)^T h_{k+1,N} - \left(RG_{k,N}\right)^T u_k^*$$
$$-\left(A - BG_{k,N}\right)^T K_{k+1,N} C x_k + Q y_k^*, \tag{3e}$$

$$h_{N,N} = Q y_N^*.$$

The solution is found by backwards iteration. First eq. (3d) is solved. Then, using this result, eq. (3e) can be solved. Once these two equations are solved, the optimal control can be computed from eqs. (3c), (3b), and (3a), respectively.

3. The optimal regulator for an infinite planning horizon

In the previous section we considered the finite planning horizon regulator. Now, we derive under some additional conditions the optimal control algorithm if the planning horizon is extended to infinity. The proof of the main theorem uses the fact that if the matrix pair (A, B) is stabilizable, the solution $K_{0,N}$ of the recurrence eq. (3d) converges when N tends to infinity. [Recall that a matrix pair (A, B) is called stabilizable if there exists a matrix F such that all eigenvalues of the matrix $A + BF$ are situated inside the unit disc.] This statement will be proved in Proposition 3. In fact, it is a by-product of the well-known infinite horizon Linear Quadratic regulator problem. Therefore, we first summarize some results in Propositions 1 and 2 which can be easily deduced from, e.g., the results obtained by Preston (1977) [see also Preston et al. (1982, sects. 11.4, 12.3) or Engwerda (1988b, sect. V.3)].

Proposition 1. *Consider the system $y_{k+1} = Ay_k + Bu_k$, $k = 0, 1, \ldots, N-1$. The control law minimizing the quadratic cost functional*

$$\bar{J}_N = y_N^T Q y_N + \sum_{k=0}^{N-1} \left(y_k^T Q y_k + u_k^T R u_k \right),$$

where Q and R are symmetric semi-positive definite matrices and $B^T Q B + R$ is a positive definite matrix, is given by

$$\bar{u}_{k,N} = -G_{k,N} y_k, \qquad k = 0, 1, \ldots, N-1.$$

Here $G_{k,N}$ is given again by eq. (3b). The minimal value of the cost functional is $y_0^T K_{0,N} y_0$.

Proposition 2. *Consider the system and cost functional stated in Proposition 1, and assume that the matrix pair (A, B) is stabilizable. Then, there is a control sequence $u(.)$ such that $\lim_{N \to \infty} \bar{J}_N$ exists.*

Proposition 3. *Let (A, B) be stabilizable and Q and R satisfy the conditions from Proposition 1. Then, $G_{k,N}$ given by the following algorithm converges to a limit independent of k when N tends to infinity:*

$$G_{k,N} = \left(B^T K_{k+1,N} B + R \right)^{-1} B^T K_{k+1,N} A,$$

where $K_{k,N}$ satisfies the Recursive Riccati Equation

$$\begin{aligned} K_{k-1,N} &= Q + A^T K_{k,N} A \\ &\qquad - A^T K_{k,N} B \left(R + B^T K_{k,N} B \right)^{-1} B^T K_{k,N} A, \qquad \text{(RRE)} \\ K_{N,N} &= Q. \end{aligned}$$

Proof. Consider once again the system and cost functional from Proposition 1. Since (A, B) is stabilizable, we have from Proposition 2 that $\lim_{N \to \infty} \bar{J}_N$ exists. Since $0 \le \bar{J}_N \le \bar{J}_{N+1}$, straightforward application of Proposition 1 yields the stated result.

The following corollary is an immediate consequence of this proposition.

Corollary 1. The limit G $(=\lim_{N\to\infty} G_{k,N})$ *satisfies the following matrix equation*:

$$G = (B^TKB + R)^{-1}B^TKA,$$

where K *is a semi-positive definite solution of the Algebraic Riccati Equation*

$$K = A^T\{K - KB(B^TKB + R)^{-1}B^TK\}A + Q. \qquad \text{(ARE)}$$

Note that the solution of (ARE) is in general not uniquely determined.

Proposition 3 gives a sufficient condition to conclude for the convergence of the solution of the recurrence eq. (RRE). This result will be used in a corollary from Theorem 2 to conclude that the optimal control $\bar{u}_{i,N}$ in (3a) converges when N tends to infinity.

In Theorem 2 we prove that, if $\bar{u}_i := \lim \bar{u}_{i,N}$ exists for all i, then this control sequence minimizes the infinite planning horizon problem. To prove this result, certain limits and summations have to be interchanged. A general result on this subject is stated in Proposition 4.

Proposition 4. Let f_N *be a sequence of positive functions (i.e.,* $f_N(x) > 0$ *for all* x*) which converges pointwise to a function* f*. Assume furthermore that* $\lim_{N\to\infty} \int f_N$ *is finite. Then,* $\int \lim_{N\to\infty} f_N \le \lim_{N\to\infty} \int f_N$.

Proof. We know that

$$\liminf \int f_N = \lim \int f_N.$$

By assumption $\lim \int f_N$ is finite, so we have that $\liminf \int f_N$ is finite. Thus the conditions for applying Fatou's lemma are satisfied. According to this lemma we have:

$$\int \liminf f_N \le \liminf \int f_N.$$

As f_N converges to f it is obvious that $\liminf f_N$ equals f. Using this, we obtain $\int \lim f_N \le \lim \int f_N$, which had to be proven.

We now arrive at the promised theorem.

Theorem 2. *Assume that in eq. (3a), for all i, $\bar{u}_i := \lim_{N\to\infty} \bar{u}_{i,N}$ exists. Then, the control sequence $\bar{u}_i$ minimizes the following infinite time horizon cost criterion*:

$$J_\infty := \lim_{N\to\infty} J_N = \lim_{N\to\infty} \sum_{k=0}^{N-1} \{(y_k - y_k^*)^T Q(y_k - y_k^*) + (u_k - u_k^*)^T R(u_k - u_k^*)\} + (y_N - y_N^*)^T Q(y_N - y_N^*), \tag{4a}$$

where $Q \geq 0$ and $R \geq 0$ are symmetric and $B^TQB + R > 0$.

Proof. Consider the optimal control sequence $u_{i,N}$ that minimizes the cost functional J_N given in Theorem 1. Denote the resulting control cost by $\bar{J}_N$. Since $\bar{J}_N$ is a monotonically increasing sequence, we have, due to Bellman's principle, that for any sequence u

(i) $\quad \lim_{N\to\infty} \bar{J}_N \leq J_\infty(u_i)$.

So the only thing left to prove is that

(ii) $\quad \lim \bar{J}_N \geq J_\infty(\bar{u}_i)$.

From inequality (i) we have that, in case $\lim \bar{J}_N$ is infinite, any control will minimize the infinite cost criterion. Therefore, we assume in the sequel, without loss of generality, that $\lim \bar{J}_N$ is finite. To prove inequality (ii), we introduce the following function:

$$f_{k,N} := (y_{k+1} - y_{k+1}^*)^T Q(y_{k+1} - y_{k+1}^*) + (\bar{u}_{k,N} - u_k^*)^T R(\bar{u}_{k,N} - u_k^*),$$

and

$$f_{k,N} := 0 \quad \text{if} \quad k > N.$$

Application of Proposition 4 yields

$$\sum_{k=0}^{\infty} \lim f_{k,N} \leq \lim \sum_{k=0}^{\infty} f_{k,N},$$

which can be rewritten as

$$\sum_{k=0}^{\infty} \lim f_{k,N} \le \lim \sum_{k=0}^{N} f_{k,N}.$$

This completes the proof.

Corollary 2. Assume that the matrix pair (A, B) in eq. (1) is stabilizable and the uncontrollable vectors $x(.)$ and reference trajectories $y^*(.)$ and $u^*(.)$ are such that, for all $i \ge 0$, $h_i := \lim_{N\to\infty} h_{i,N}$ exists. Then, the optimal control minimizing J_∞ [eq. (4a)] is given by

$$\bar{u}_i = -(R + B^TKB)^{-1}B^TKAy_i - (R + B^TKB)^{-1}B^T\{KCx_i - h_{i+1}\}$$
$$+ (R + B^TKB)^{-1}Ru_i^*, \tag{4b}$$

where K is the limit of the recurrence relation given by (RRE).

Proof. According to Theorem 1, the optimal control for the finite horizon problem is given by

$$\bar{u}_{k,N} = -G_{k,N}y_k - g_{k,N},$$

where $G_{k,N}$ and $g_{k,N}$ satisfy eqs. (3b) and (3c).

Since (A, B) is stabilizable we have, according to Proposition 3, that $K_{k+1,N}$ converges when N tends to infinity. This, together with the assumption that $\lim h_{k,N}$ exists for all k, leads to the conclusion that $\lim \bar{u}_{k,N}$ converges to $\bar{u}_k$, where $\bar{u}_k$ is given by eq. (4b). So, for all $i \ge 0$, $\lim \bar{u}_{i,N}$ exists.

Application of Theorem 2 yields the stated result.

This corollary gives rise to the next algorithm for calculating the optimal control.

Algorithm. The following steps have to be taken successively in order to calculate the optimal control:

1) *Check whether the matrix pair (A, B) is stabilizable and Q, R, and $B^TQB + R$ are symmetric semi-positive definite, respectively, positive definite matrices.*

2) *Calculate the semi-positive definite solutions of (RRE) and their limit.*

3) *Check if the exogenous and reference trajectories are such that* $\lim h_{i+1,N}$ *exists for all i.*

4) *Implement the optimal control*

$$u_i = -(R + B^T KB)^{-1} B^T KAy_i - (R + B^T KB)^{-1} B^T \{ KCx_i - h_{i+1} \}$$

$$+ (R + B^T KB)^{-1} Ru_i^*.$$

Since steps 2 and 3 are difficult to calculate, it would be nice if some additional conditions could be formulated under which these calculations become easier. Conditions which simplify the calculation of (ARE) will be discussed in the remainder of this section. Section 4 will treat conditions which ascertain the existence of $\lim_{N \to \infty} h_{i,N}$. We already note here that, in case all eigenvalues of matrix $A - BG$ differ from zero and the existence of h_i is guaranteed for all i, calculation of h_i becomes much simpler. Then h_{i+1} can be derived from h_i in the following way:

$$h_{i+1} := \left\{ (A - BG)^T \right\}^{-1} \left\{ h_i - Qy_i^* + (RG)^T u_i^* \right\} + KCx_i.$$

We shall now discuss conditions which simplify the calculation of (RRE). To that end we state first a proposition which gives sufficient conditions to conclude that all eigenvalues of matrix $A - BG$ are situated inside the unit circle. Using this result we prove then in Proposition 6 that eq. (ARE) possesses a unique positive definite solution.

Proposition 5. Assume that (A, B) is stabilizable, and let matrix Q be factorized as $C^T C$. Then, all eigenvalues of matrix $A - BG$ are situated inside the unit circle if either one of the following conditions is satisfied:

(i) *R is positive definite and the system $y_{k+1} = Ay_k + Bu_k$; $z = Cy$ is detectable.*

(ii) *Q is positive definite and R is semi-positive definite.*

Proof. In case condition (i) is satisfied, the proof can be found, e.g., in Wonham (1979, theorems 3.6, 12.2). If condition (ii) is satisfied, the proof follows directly from the fact that (ARE) can be rewritten as

$$K = (A - BG)^T K (A - BG) + Q + G^T RG.$$

Since $Q + G^T RG$ is a positive definite matrix, it is not difficult to prove that all eigenvalues of matrix $A - BG$ are in norm smaller than one [see, e.g., Chow (1975, p. 175, problem 8)].

Proposition 6. *Assume that (A, B) is stabilizable. If either condition (i) or (ii) in Proposition 5 is satisfied, then (ARE) has a unique semi-positive definite solution. In case condition (ii) is satisfied, this solution is positive definite.*

Proof. First, note that the matrix equation $F^TPF - P + Q = 0$ possesses a unique semi-positive definite solution P if Q is semi-positive definite and all eigenvalues of matrix F are in norm smaller than one. Using this, together with the result obtained in Proposition 5, proves the first part of the proposition. That condition (ii) even implies that the solution of (ARE) will be positive definite follows immediately from the observation that $K \geq Q + G^TRG$ (see Proposition 5).

The advantage of this last proposition is that, under the assumptions mentioned, we only have to solve one quadratic matrix equation from which we know that it has exactly one solution in the class of semi-positive definite matrices.

4. Additional conditions simplifying the algorithm

The main topic of this section will be the derivation of conditions under which the existence of $\lim_{N\to\infty} h_{i,N}$ is ascertained. The conditions we shall give will prove to be rather weak.

Moreover, we give some (rather weak) assumptions on the system which guarantee the invertibility of matrix $A - BG$. This simplifies the calculation of h_{i+1} from h_i (see section 3) considerably. The consequence of this is that both Theorem 2 and the algorithm, derived in section 3, can be simplified. Therefore, we reformulate them in case these additional assumptions hold. We start this section with stating the basic result of this section (and paper). It concerns the existence of $\lim_{N\to\infty} h_{i,N}$. The next theorem gives sufficient conditions for the existence of this limit. The proof of it is rather technical and depends strongly on a result Aulbach derived in (1984, lemma B5, corollary B5). It is therefore deferred to appendix 1, where also Aulbach's lemma is stated.

Theorem 3. *Assume that* $\max|\sigma(A - BG)| = l_1 < 1$ *and $K_{k,N}$ converges in (RRE). Then,* $\lim_{N\to\infty} h_{k,N}$ *exists if the growth rate of the exogenous and reference values is smaller than $1/l_1$, i.e., $\|y^*_{k+1}\| \leq m_1\|y^*_k\|$, $\|u^*_{k+1}\| \leq m_1\|u^*_k\|$, and $\|x_{k+1}\| \leq m_1\|x_k\|$, where $1/m_1 > l_1$, for all k. Moreover $h_{k,N} \to h_k$, where h_k is defined by*

$$h_k = \sum_{i=k}^{\infty} \left\{(A - BG)^T\right\}^{i-k} \left\{Qy_i^* - (RG)^T u_i^* - (A - BG)^T KCx_i\right\}. \tag{5}$$

The promised important special case of Theorem 2 then reads as follows:

Theorem 4. Assume that one of the conditions of Proposition 5 is satisfied, and that the pair (A, B) is stabilizable. Denote $\max|\sigma(A - BG)|$ *by* l_1. *Then, under the additional assumption that the growth rate of the reference and exogenous deterministic variables is smaller than* $1/l_1$, *the optimal control minimizing* $\lim J_N$ *(see Theorem 2) is given by (4b), where* K *is the unique semi-positive definite solution of (ARE) and* h_k *is given by (5).*

Proof. According to Proposition 5, all eigenvalues of matrix $A - BG$ are situated inside the unit circle. This implies, according to Theorem 3, that $h_{k,N}$ will converge for all k to h_k, which is given by eq. (5). Corollary 2 yields then the stated result.

From this theorem it is clear that the influence of future reference and exogenous variables on the optimal control is exponentially decreasing. So, provided the conditions stated in the theorem are all satisfied, the specification of the last part of the reference trajectories becomes superfluous. Of course this makes application of the controller in practice easier. The disadvantage of this controller is that, at any time k, h_k must be calculated as an infinite sum. In the following theorem we give sufficient conditions which make it possible to calculate h_{k+1} from h_k, once we have calculated an initial value, say h_1.

Theorem 5. Assume that $Q > 0$, $R > 0$, *matrices* A *and* B *are full column rank, and the pair* (A, B) *is stabilizable. Denote* $\max|\sigma(A - BG)|$ *by* l_1. *Then, under the additional assumption that the growth rate of the reference and exogenous variables is smaller than* $1/l_1$, *the optimal control minimizing* $\lim J_N$ *(see Theorem 2) is given by (4b), where* K *is the positive definite solution of (ARE) and* h_k *is given by the following recurrence equation:*

$$h_1 = \sum_{i=1}^{\infty} \{(A - BG)^T\}^{i-1}\{Qy_i^* - (RG)^T u_i^* - (A - BG)^T KCx_i\},$$

$$h_{k+1} = \{(A - BG)^T\}^{-1}\{h_k - Qy_k^* + (RG)^T u_k^*\} + KCx_k. \tag{6}$$

Proof. From Proposition 6 we have that under these conditions K is positive definite. This implies, using the assumption that $R > 0$ and matrices A and B are full column rank, that matrix $A - BG$ is invertible.

In the proof of Theorem 3 we already noted that h_i satisfies the recurrence equation

$$h_i = (A - BG)^T h_{i+1} + Qy_i^* - (RG)^T u_i^* - (A - BG)^T KCx_i,$$

$$h_1 = \sum_{i=1}^{\infty} \{(A - BG)^T\}^{i-1} \{Qy_i^* - (RG)^T u_i^* - (A - BG)^T KCx_i\}.$$

Using the invertibility of $A - BG$ and Theorem 4, we obtain the stated result.

Provided the assumptions stated in Theorem 5 are satisfied, important simplifications in the algorithm for calculating the optimal control are possible. We conclude this section with a reformulation of the algorithm under those conditions.

Simplified algorithm resulting from Theorem 5

1a) *Check whether (i) $Q > 0$, $R > 0$, and (ii) A and B are full column rank.*
b) *Check whether (A, B) is stabilizable.*

2a) *Calculate the positive definite solution of (ARE).*
b) *Calculate* $\max|\sigma(A - BG)| = l_1$.

3a) *Check whether the growth rate of the exogenous and reference variables is not exceeding* $1/l_1$.
b) *Calculate h_1.*

4a) *Implement the optimal control given by eq. (4b).*
b) *Calculate h_{i+1} from eq. (6).*
c) *Increment i by* 1 *and return to step 4a.*

5. The infinite time Minimum Variance and Linear Quadratic regulator

In this section we consider two special cases of the optimal controller we derived in section 3. Furthermore we give a characterization of the admissible reference trajectories for system (1).

The first special case we consider is the controller minimizing the infinite horizon Minimum Variance cost criterion. Chow (1975, sect. 7.8) already derived under certain conditions this controller. He was, however, not able to check all these conditions beforehand. We will show now that his assumption concerning G, namely that $A - BG$ has its eigenvalues inside the unit circle, is always satisfied provided that matrix (A, B) is stabilizable. In Corollary 3 we state the exact conditions under which the regulator is optimal.

Corollary 3. Suppose that the matrix pair (A, B) is stabilizable and that $\lim h_{k+1,N}$ is finite. Let y_k satisfy system (1) and let Q be symmetric, positive definite. Then, the regulator minimizing the cost criterion,

$$\lim_{N\to\infty} \sum_{i=0}^{N} (y_i - y_i^*)^T Q(y_i - y_i^*),$$

is given by

$$u_k = -Gy_k - g_k,$$

where $G = (B^TKB)^{-1}B^TKA$, with K the positive definite solution of

$$K = A^T\left\{K - KB(B^TKB)^{-1}B^TK\right\}A + Q$$

and

$$g_k = (B^TKB)^{-1}B^T\left(KCx_k - \sum_{i=k+1}^{\infty} \left\{(A-BG)^T\right\}^{i-k-1} \times \left(Qy_i^* - (A-BG)^TKCx_i\right)\right).$$

Proof. Take $R = 0$ in Theorem 4.

Notice that, in case matrix B is invertible, the solution of (ARE) is matrix Q and the resulting optimal control equals the Minimum Variance control. Another interesting point is that, by taking $y_j^* = 0$ for $j \geq k+1$ and $x_j = 0$ for $j \geq k+1$, we get a stabilizing Minimum Variance controller for a zero set point.

The second special case of the optimal controller derived in section 3 is the Linear Quadratic regulator. This one is obtained by taking all reference and exogenous variables equal to zero in Theorem 2 [see, e.g., Bertsekas (1976, sect. 3.1)]. Having deduced the infinite planning horizon controller, we can give an exact characterization of all reference trajectories that can be tracked ultimately by system (1).

To this extent we first define what we mean by this.

Definition 1. A reference trajectory y_k^* is called asymptotically admissible for $\bar{y}_0$ if there exists a control sequence u_k such that $\|y_k - y_k^*\| \to 0$ for $k \to \infty$. In that case u_k is called a successful control.

Theorem 6. Consider system (1). Assume that (A, B) is stabilizable. Then, a reference trajectory is asymptotically admissible if and only if there exist u_k^* and

v_k such that

$$y^*_{k+1} = Ay^*_k + Bu^*_k + Cx_k + v_k; \qquad y_0^* = y, \tag{7}$$

where $v_k \to 0$. Moreover, the following controller is successful:

$$u_k = u^*_k - G(y_k - y^*_k) + (R + B^TKB)^{-1}B^TK(v_i + h_{i+1}), \tag{8}$$

where $G = (I + B^TKB)^{-1}KA$, K is the positive definite solution of

$$K = A^T\{K - KB(I + B^TKB)^{-1}B^TK\}A + I$$

and

$$h_k = \sum_{i=k}^{\infty} \{(A - BG)^T\}^{i-k+1} v_i.$$

Proof. We first show that an asymptotic admissible reference trajectory must satisfy eq. (7). Note that the sequence $v'_k := y^*_k - y_k$ tends to zero. Consequently,

$$\begin{aligned} y^*_{k+1} &= y_{k+1} + v'_{k+1} \\ &= Ay_k + Bu^*_k + Cx_k + v'_{k+1} \\ &= A(y^*_k - v'_k) + Bu^*_k + Cx_k + v'_{k+1} \\ &= Ay^*_k + Bu^*_k + Cx_k + v_k, \end{aligned}$$

where v_k equals $v'_{k+1} - Av'_k$. This proves one part of the claim.

We show now that any reference trajectory satisfying eq. (7) is tracked by controller (8). To that end, we consider the following minimization problem:

$$\min_{u(.)} \lim_{N\to\infty} \sum_{k=0}^{N-1} \{\|y_k - y^*_k\|^2 + \|u_k - u^*_k\|^2\} + \|y_N - y^*_N\|^2,$$

subject to system (1).

Introducing the new variables $y'_k = y_k - y^*_k$ and $u'_k = u_k - u^*_k$, we see that this problem can be rewritten as

$$\min_{u'(.)} \lim_{N\to\infty} \sum_{k=0}^{N-1} \{\|y'_k\|^2 + \|u'_k\|^2\} + \|y'_N\|^2,$$

subject to Σ': $y'_{k+1} = Ay'_k + Bu'_k - v_k$.

Since this minimization problem satisfies all conditions posed in Theorem 4, control algorithm (4b) yields the optimal controller. Substitution of the model parameters into (4b) gives then rise to control algorithm (8).

Moreover, we have that the closed-loop system can be rewritten as

$$y'_{k+1} = (A - BG)y'_k + B(R + B^TKB)^{-1}B^TK(v_k + h_{k+1}),$$

where $A - BG$ is exponentially stable and v_k, h_{k+1} converge to zero. Therefore, we conclude that y'_k converges to zero when k tends to infinity. This completes the proof.

We conclude this section by noting that an exact characterization of asymptotically admissible reference trajectories for more general systems than (1) can be found in Engwerda (1988b).

6. Conclusions

In this paper we derived an optimal control algorithm minimizing the infinite planning horizon quadratic tracking cost functional for a linear system possessing an exogenous uncontrollable component. The algorithm is obtained under a restriction on the exogenous and reference trajectories, the condition that a weighted sum of the weighting matrices is positive definite, and the assumption that the system without the exogenous component is stabilizable.

Application of the algorithm in its most general form proves to be cumbersome. Too many calculations are needed. Therefore, under some (weak) additional conditions, a special case of the algorithm is derived. This last algorithm proves to be easy to implement. An important part of the calculations, the determination of the Riccati equation, can be done off-line. Moreover, only one equation has to be updated for the determination of the optimal control.

A disadvantage of this control scheme is that all exogenous and reference trajectories should be known in advance over an infinite time horizon. By imposing a weak restriction on the weighting matrix Q it is shown that this disadvantage can partly be overcome. In that case the future exogenous and reference values are exponentially weighted by a matrix that has all its eigenvalues inside the unit circle. As a consequence the influence of the future variables on the control decays exponentially fast. Therefore exact specification of the future reference and exogenous paths becomes less demanding to implement a nearby optimal controller.

A special case of the algorithm is the infinite time Minimum Variance controller. An advantage of this controller relative to the (one timestep)

Minimum Variance controller is, that now BIBO stability of the closed-loop system is always achieved.

Finally, we noted that the Linear Quadratic regulator can also be obtained from this controller, and that this controller can be used to track any asymptotically admissible reference trajectory in a stabilizable system.

Appendix

Before we give the proof of Theorem 3, we first quote Aulbach's perturbation lemma.

Lemma (Aulbach's perturbation lemma). *Consider the homogeneous difference equation*

$$x_{k+1} = A_k x_k$$

and a perturbation

$$x_{k+1} = (A_k + B_k) x_k, \tag{9}$$

whose respective principal fundamental matrices are denoted by $\Phi(k, l)$ and $\Psi(k, l)$. Suppose that eq. (9) is defined for all k from a set J of consecutive integers. Furthermore suppose that $\|B_k\|$ is bounded above on J by some positive constant δ. Then, the following is true: If there exist positive constants γ, λ such that

$$\|\Phi(k, l)\| \le \gamma\lambda^{k-l} \quad \text{for all} \quad k, l \in J, \quad k \ge l+1,$$

then

$$\|\Psi(k, l)\| \le \beta\mu^{k-l} \quad \text{for all} \quad k, l \in J, \quad k \ge l+1,$$

with

$$\beta := (\gamma\lambda + \gamma\delta)/(\lambda + \gamma\delta) \quad \text{and} \quad \mu := \lambda + \gamma\delta.$$

The proof of Theorem 3 now reads as follows:

Proof of Theorem 3. Consider the following two homogeneous difference equations

$$z_{k+1} = (A - BG)^T z_k \quad \text{and} \quad z_{k+1} = \left((A - BG)^T + E_k\right) z_k,$$

where $E_k = (A - BG_{N-k,N})^T - (A - BG)^T$. Denote the principal fundamental matrix of the second equation by $X(k, l)$, with $X(k, k) = I$ and $X(k, l) = 0$ for $l > k$. Note that E_k does not depend on N, due to the fact that $G_{N-k,N}$ depends only on the difference between N and $N - k$ [see eq. (3d)]. Since $\max|\sigma(A - BG)| = l_1 < 1$, there exists for all $l_2 > l_1$ a constant $M(l_2)$ such that $\|(A - BG)^n\| \le M(l_2) * l_2^n$.

As E_k converges to zero when k tends to infinity it is clear now, from Aulbach's lemma, that for any $1/m_1 > l_1$ there exists a k_1 and M such that for all $k \ge k_1$ and $l \ge k + 1$, $\|X(k, l)\| \le M * r_1^{k-l}$.

Straightforward calculation shows that $h_{k,N}$ equals

$$\text{(i)} \qquad X(N-k,0)\,Qy_N^* + \sum_{j=0}^{N-k-1} X(N-k,\, N-k-j)\, v_{k+j,N},$$

where

$$v_{j,N} = -\left(RG_{j,N}\right)^T u_j^* - \left(A - BG_{j,N}\right)^T K_{j+1,N} C x_j + Qy_j^*.$$

In the following we shall prove that $h_{k,N}$ is a Cauchy sequence. Since $\mathbb{R}^n$ is complete, we can conclude then that $\lim_{N\to\infty} h_{k,N}$ exists. In the proof we will need the next two properties:

$$\text{1)} \qquad X(N-k,0)\,Qy_N^* \to 0 \quad \text{when} \quad N \to \infty,$$

$$\text{2)} \qquad \left\| \sum_{j=k_2}^{N-k-1} X(N-k,\, N-k-j)\, v_{k+j,N} \right\| \le M_2 * (r_1 * m_1)^{k_2},$$

where M_2 is a constant independent of N and k, and $0 < r_1 < 1/m_1$.

These two properties will be proved first. Due to the assumption of bounded exponential growth for the exogenous and reference variables and the fact that all matrices are bounded in norm, it is immediately clear that the following inequalities hold for some constants P_1 and P_2:

$$\text{(ii)} \qquad \|v_{j,N}\| \le P_1 * m_1^{j-k},$$

$$\begin{aligned}\text{(iii)} \qquad \|X(N-k,0)\,Qy_N^*\| &= \|X(N-k,\, k_1)\, X(k_1,0)\,Qy_N^*\| \\ &\le \|X(N-k,\, k_1)\|\,\|X(k_1,0)\,Qy_N^*\| \\ &\le M r_1^{N-k-k_1} * P_2 m_1^{N-k}.\end{aligned}$$

Since $r_1 < m_1 < 1$ and k_1 is finite, property 1) results immediately from the last inequality (iii).

To prove property 2) we note furthermore that

$$\sum_{j=k_2}^{N-k-1} X(N-k, N-k-j) v_{k+j,N}$$

$$= \sum_{j=k_2}^{N-k-k_1} X(N-k, N-k-j) v_{k+j,N}$$

$$+ \sum_{j=N-k-k_1+1}^{N-k-1} X(N-k, k_1) X(k_1, N-k-j) v_{k+j,N},$$

so that in the same way, using (ii), we have

$$\left\| \sum_{j-k_2}^{N-k-1} X(N-k, N-k-j) v_{j+k_2,N} \right\|$$

$$\leq \sum_{j=k_2}^{N-k-k_1} M r_1^j P_1 m_1^j + \sum_{j=N-k-k_1+k}^{N-k-1} M r_1^{N-k-k_1} P_3 m_1^j.$$

Now the second term,

$$\sum_{j=N-k-k_1+1}^{N-k-1} M r_1^{N-k-k_1} P_3 m_1^j$$

$$\leq (r_1 * m_1)^{N-k-k_1} \sum_{j=N-k-k_1+1}^{N-k-1} M P_3 m_1^{j-N+k+k_1}$$

$$= (r_1 * m_1)^{N-k-k_1} P_4,$$

where P_4 is a constant independent of N and k, and the first term

$$\sum_{j=k_2}^{N-k-k_1} M P_1 (r_1 * m_1)^j = M P_1 (r_1 * m_1)^{k_2} \sum_{j=k_2}^{N-k-k_1} (r_1 * m_1)^{j-k_2}$$

$$= M P_1 P_5 (N-k-k_1)(r_1 * m_1)^{k_2},$$

where $P_5(N-k-k_1)$ is a constant depending on $N-k-k_1$. Since the constant P_5 is bounded ($r_1 * m_1 < 1!$), we can conclude now that there exists a constant M_2, independent of N, k and k_2 such that 2) holds.

We shall prove now that $h_{k,N}$ is a Cauchy sequence. Therefore consider

$$h_k(N, m) := h_{k,N+m} - h_{k,N}.$$

Substitution of (i) yields

$$\begin{aligned} h_k(N, m) = {} & INIT \\ & + \sum_{j=N-k}^{N+m-k-1} X(N+m-k, N+m-k-j) v_{k+j,N+m} \\ & + \sum_{j=0}^{N-k-1} \{ X(N+m-k, N+m-k-j) v_{k+j,N+m} \\ & - X(N-k, N-k-j) v_{k+j,N} \}, \end{aligned}$$

where

$$INIT = X(N+m-k, 0) Q y^*_{N+m} - X(N-k, 0) Q y^*_N.$$

From 1), respectively 2), it is immediately clear that $INIT$ and $\sum_{j=N-k}^{N+m-k-1} X(N+m-k, N+m-k-j) v_{k+j,N+m}$ converge to zero when N tends to infinity. Consequently it suffices to prove now that the third term of $h_k(N, m)$, $\sum_{j=0}^{N-k-1} \{ X(N+m-k, N+m-k-j) v_{k+j,N+m} - X(N-k, N-k-j) v_{k+j,N} \}$, becomes in norm arbitrarily small when N tends to infinity. To obtain this result, we note that this sum can be rewritten as

$$\begin{aligned} v(N) := {} & \sum_{j=0}^{N-k-1} X(N+m-k, N+m-k-j)(v_{k+j,N+m} - v_{k+j,N}) \\ & + \sum_{j=0}^{N-k-1} [X(N+m-k, N+m-k-j) \\ & - X(N-k, N-k-j)] v_{k+j,N}. \end{aligned}$$

Since, for any j, $v_{j,N}$ converges when N tends to infinity, we can conclude that, for any $eps > 0$ and for all N, there exists a k_3 such that for all $j \leq k_3 \| v_{j,N+m} - v_{j,N} \| < eps$, and moreover k_3 tends to infinity when N tends to infinity. Using this property, we can estimate the first term of $v(N)$ by

$$eps \sum_{j=0}^{k_3} \| X(N+m-k, N+m-k-j) \|$$

$$+ \sum_{j=k_3+1}^{N-k-1} \| X(N+m-k, N+m-k-j) \| \| v_{k+j,N+m} - v_{k+j,N} \|.$$

As $\lim_{N \to \infty} \sum_{j=0}^{N-k-1} \| X(N+m-k, N+m-k-j) \|$ exists, it is clear that $epsX(N+m-k, N+m-k-j)$ can be made arbitrarily small by an appropriate choice of eps. Due to the monotonic dependency of k_3 on N, from 2) it is obvious that

$$\sum_{j=k_3+1}^{N-k-1} \| X(N+m-k, N+m-k-j) \| \| v_{k+j,N+m} - v_{k+j,N} \|$$

will become arbitrarily small too, if N is chosen large enough. This completes the proof of the convergence of the first part of $v(N)$ to zero. For the second part, $\sum_{j=0}^{N-k-1} \{ X(N+m-k, N+m-k-j) - X(N-k, N-k-j)] v_{k+j,N}$, a similar argument is used. In this case we have that, for all $eps > 0$ and all integers N, there exists a k_3 such that for all $j \leq k_3 \| X(N+m-k, N+m-k-j) - (A-BG)^j \| < eps$, where again k_3 tends to infinity if N does so. By splitting up this sum, as was done for the first part of $v(N)$, it is seen that this term converges also to zero when N tends to infinity.

This completes the proof that $h_k(N, m)$ is a Cauchy sequence. So we have proved now, that $\lim h_{k,N}$ exists.

The second statement of the proposition is that $h_{k,N} \to h_k$, where h_k is given by (5). Note that it is not difficult, under the assumptions stated in the proposition, to prove that h_k exists for any k. Once we can prove now that $h_{0,N} \to h_0$, it is seen via the following reasoning that $h_{k,N}$ converges then to h_k when N tends to infinity. From eq. (3e) we have that $\lim h_{k,N}$, denote it by $\bar{h}_k$, satisfies the recurrence equation

$$\bar{h}_k = (A - BG) \bar{h}_{k+1} + v_k,$$

with

$$v_k = -(RG)^T u_k^* - (A - BG)^T KCx_k + Qy_k^*.$$

It is now easily verified that h_k satisfies this recurrence equation too. So, once we proved that h_0 and $\bar{h}_0$ coincide, we can conclude then that h_k and $\bar{h}_k$ are identical for any k. The proof that $h_{0,N} \to h_0$ reads as follows:

$$\lim \|h_{0,N} - h_0\|$$

$$= \lim \left\| X(N,0)Qy_0^* + \sum_{j=0}^{N-1} \left[X(N, N-j) v_{j,N} - \{(A - BG)^T\}^j v_j \right] \right.$$

$$\left. + \sum_{j=N}^{\infty} \{(A - BG)^T\}^j v_j \right\|$$

$$\leq \lim \| X(N,0)Qy_0^* \|$$

$$+ \lim \left\| \sum_{j=0}^{N-1} \left[X(N, N-j) v_{j,N} - \{(A - BG)^T\}^j v_j \right] \right\|$$

$$+ \lim \left\| \sum_{j=N}^{\infty} (A - BG)^T\}^j v_j \right\|.$$

Due to respectively 1), the analysis of the second term of $v(N)$, and 2) it is immediately clear that each of these terms converges to zero when N tends to infinity. So we can conclude that $\lim h_{0,N} = h_0$, which completes the proof of Theorem 3.

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